

MALLIAVIN CALCULUS METHOD FOR ASYMPTOTIC EXPANSION OF DUAL CONTROL PROBLEMS

MICHAEL MONOYIOS
MATHEMATICAL INSTITUTE
UNIVERSITY OF OXFORD

ABSTRACT. We develop a technique based on Malliavin calculus ideas, for asymptotic expansion of dual control problems arising in connection with exponential indifference valuation of claims, and with minimisation of relative entropy, in incomplete markets. The problems involve optimisation of a functional in which the control features quadratically, while in the state dynamics it appears as a drift perturbation to Brownian paths on Wiener space. This drift is interpreted as a measure change using the Girsanov theorem, leading to a form of the integration by parts formula in which a directional derivative on Wiener space is computed. This allows for asymptotic analysis of the control problem. Applications to incomplete Itô process markets are given, in which indifference prices are approximated in the low risk aversion limit. We also give an application to identifying the minimal entropy martingale measure as a perturbation to the minimal martingale measure in stochastic volatility models.

1. INTRODUCTION

In this article we use an approach to the Malliavin calculus, in which perturbations to Brownian paths on Wiener space are interpreted as measure changes via the Girsanov theorem, to derive asymptotic expansions of stochastic control problems which arise from the dual to investment and indifference pricing problems under exponential utility.

In the dual approach to investment and hedging problems in incomplete markets, optimisation problems over trading strategies are converted to optimisations over probability measures. For example, in exponential indifference pricing of a claim with payoff F , we shall show (Lemma 4.8) that the dual control representation of the time-zero indifference price is

$$(1.1) \quad p_0 = \sup_{\mathbb{Q} \in \mathbf{M}_f} \left[\mathbb{E}^{\mathbb{Q}}[F] - \frac{1}{\alpha} I_0(\mathbb{Q}|\mathbb{Q}^0) \right],$$

where \mathbf{M}_f is a space of equivalent local martingale measures (ELMMs), $\alpha > 0$ is the risk aversion coefficient, \mathbb{Q}^0 is the minimal entropy martingale measure (MEMM) and $I_0(\mathbb{Q}|\mathbb{Q}^0)$ is the relative entropy between any ELMM \mathbb{Q} and \mathbb{Q}^0 . In an Itô process setting, the optimisation over measures in (1.1) converts to a problem in which the control is a drift perturbation to a multi-dimensional Brownian motion. One views this drift as a perturbation to Brownian paths on Wiener space, and then Malliavin calculus ideas arise in deriving an asymptotic expansion for the indifference price, valid for small α . Similar ideas arise in entropy minimisation problems, which are the dual to pure investment problems with exponential utility, and we shall illustrate an example of this in a stochastic volatility model, in which the small parameter is $1 - \rho^2$, ρ being the correlation between the stock and its volatility.

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Utility-based valuation techniques rarely lead to explicit solutions, and this motivates the interest in approximate solutions. The idea of using Malliavin calculus methods in asymptotic indifference pricing is due to Davis [7]. Davis used the approach in a two-dimensional constant parameter basis risk model, with a traded and non-traded asset following correlated geometric Brownian motions, and for a European claim depending only on the final value of the non-traded asset price. In this model, it also turns out that PDE techniques, based on a Cole-Hopf transform applied to the the Hamilton-Jacobi-Bellman equation of the underlying utility maximisation problem (see Zariphopoulou [35], Henderson [15] and Monoyios [25]), leads to a closed form non-linear expectation representation for the indifference price. The asymptotic expansion obtained by Davis [7] can therefore also be obtained by applying a Taylor expansion to the non-linear expectation representation, as carried out in Monoyios [24, 27]. For this reason, perhaps, the technique developed by Davis has not been further exploited.

In higher-dimensional models, and in almost all models with random parameters, the aforementioned Cole-Hopf transform does not work. Indifference prices and their risk-aversion asymptotics have been analysed via other methods, notably BSDE and BMO methods (Mania and Schweizer [23], Kallsen and Rheinländer [20]) for bounded claims. Monoyios [28] derived small risk aversion valuation and hedging results via PDE techniques, in a random parameter basis risk model generated by incomplete information on asset drifts. Delbaen *et al* [8] and Stricker [34] used arguments based on a Fenchel inequality to derive the zero risk aversion limit of the indifference price. Recently, Henderson and Liang [16] have used BSDE and PDE techniques to derive indifference price approximations, of a different nature to ours, in a multi-dimensional non-traded assets model.

The techniques in this paper are different. We resurrect the method suggested by Davis [7]. Our contribution is first to show that this technique can be significantly generalised, to cover multi-dimensional Itô process markets, with no Markov structure required, and for claims which can be quite general functionals of the paths of the asset prices. In doing this we elucidate the precise relation with the Malliavin calculus in this general set-up. The second contribution is establish a dual stochastic control representation (Lemma 4.8) of the indifference price process in a semi-martingale model. This result seems to be the most compact representation possible. We then apply the Malliavin asymptotic method to this control problem in an Itô process setting, and derive the general form of the small risk aversion asymptotic expansion of an exponential indifference price, recovering the well-known connection between small risk aversion exponential indifference valuation and quadratic risk minimisation. Examples are given of multi-asset basis risk models, and of stochastic correlation in basis risk. Finally, we show how the technique can be applied to identify the MEMM $\mathbb{Q}^0 \equiv \mathbb{Q}_E$ as a perturbation to the minimal martingale measure \mathbb{Q}_M in a stochastic volatility model, when the stock and volatility are highly correlated.

Other types of asymptotic expansion for marginal utility-based prices, in terms of a small holding of claims, have been obtained by Kramkov and Sîrbu [21] and by Kallsen *et al* [19]. These works use utility functions defined on the positive half-line, in contrast to the exponential utility function used throughout in this paper. In stochastic volatility models, Sircar and Zariphopoulou [33] obtain asymptotic expansions for exponential indifference prices using the fast mean-reversion property of the volatility process: this approach has been significantly exploited in many scenarios (see Sircar *et al* [11]), and is of a different nature to our approach.

Malliavin calculus has found application in other areas of mathematical finance, such as models of insider trading [17], to the computation of sensitivity parameters [12], and to other forms of asymptotic expansion [3], involving sensitivity with respect to initial conditions, or with respect to perturbations of drift and diffusion coefficients, or to a parameter appearing in an expectation, as opposed to a control.

The rest of the paper is organised as follows. In Section 2 we prove a version of the Malliavin integration-by-parts formula on Wiener space, giving the directional derivative of a Brownian functional. This is applied to the asymptotic analysis of a control problem in Section 3, in which the control is over a parametrised family $\mathbb{Q}(\epsilon)$ of measures, and ϵ is a small parameter. This leads to the main result (Theorem 3.2) and illustrates the interplay between directional derivatives on Wiener space, the Malliavin derivative, and perturbation analysis. In Section 4 we derive, in a locally bounded semi-martingale model, the dual stochastic control representation of the indifference price process (Lemma 4.8) that forms the basis of the control problems we are interested in. In Section 5 we apply the asymptotic analysis of indifference valuation in an Itô process setting. In Section 6 we give examples of approximate indifference valuation in some basis risk models, and we show how the MEMM can be identified as a perturbation to the minimal martingale measure in a stochastic volatility model.

2. DIRECTIONAL DERIVATIVES OF BROWNIAN FUNCTIONALS ON WIENER SPACE

In this section we present the underlying idea in an abstract setting. We consider perturbations to Brownian paths, and the ensuing directional derivatives, on Wiener space. This is Bismut's [5] approach to the Malliavin calculus, and will be used in asymptotic analysis of control problems in the next section. In this approach, one deduces a certain invariance principle (see (2.6)) by using the Girsanov theorem to translate a drift perturbation to a Brownian motion into a change of probability measure. This approach is discussed in Chapter VIII.2 of Revuz and Yor [30], for example. Nualart [29] is a general treatise on Malliavin calculus.

The general set-up involves a Gaussian space, that is, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ plus a Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. Typically, we have a concrete representation of H as an L^2 space: $H = L^2(\mathcal{T}, \mathcal{B}, \mu)$, where $(\mathcal{T}, \mathcal{B})$ is a measurable space and μ is a σ -finite atomless measure. Define a linear isometry $\mathbb{W} : H \rightarrow L^2[(\Omega, \mathcal{F}, \mathbb{P}); \mathbb{R}]$ such that $\mathbb{W} = (\mathbb{W}(h))_{h \in H}$ is a centred Gaussian family of random variables with

$$\mathbb{E}[\mathbb{W}(h)\mathbb{W}(g)] = \langle h, g \rangle_H, \quad h, g \in H.$$

We say that \mathbb{W} is an isonormal Gaussian process.

Let $C_p^\infty(\mathbb{R}^n)$ denote the set of functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with derivatives of all orders of polynomial growth. For an isonormal process \mathbb{W} , the set \mathcal{S}_p of smooth random variables consists of elements of the form

$$F := f(\mathbb{W}(h_1), \dots, \mathbb{W}(h_n)), \quad h_i \in H, \quad i = 1, \dots, n,$$

for $f \in C_p^\infty(\mathbb{R}^n)$.

Let $L^2[(\Omega, \mathcal{F}, \mathbb{P}); H] \equiv L^2(\Omega, \mathcal{F}, \mathbb{P}) \otimes H$ denote the set of square-integrable H -valued random variables. The abstract Malliavin derivative of a random variable $F \in \mathcal{S}_p$, is an H -valued random variable $DF \in L^2[(\Omega, \mathcal{F}, \mathbb{P}); H]$ that captures a notion of differentiation of F with respect to the chance parameter ω , such that $\langle DF, h \rangle_H$ has properties of a directional derivative.

For $F \in \mathcal{S}_p$, the Malliavin derivative is defined by

$$DF := \sum_{i=1}^n \frac{\partial}{\partial x_i} f(\mathbb{W}(h_1), \dots, \mathbb{W}(h_n)) \otimes h_i.$$

This definition is not dependent on the particular concrete representation of the random variable F . When we use a concrete representation $H = L^2[(\mathcal{T}, \mathcal{B}, \mu)]$ then $L^2(\Omega, \mathcal{F}, \mathbb{P}) \otimes H = L^2[(\Omega \times \mathcal{T}, \mathcal{F} \otimes \mathcal{B}, \mathbb{P} \otimes \mu)]$, and we obtain a measurable process $(D_t F)_{t \in \mathcal{T}}$ as Malliavin derivative.

If F possesses a Malliavin derivative, we have the following well-known integration-by-parts formula (see Lemma 1.2.1 in Nualart [29]):

$$(2.1) \quad \mathbb{E}[\langle DF, h \rangle_H] = \mathbb{E}[F \mathbb{W}(h)].$$

One consequence of (2.1) is that the operator $D : \mathcal{S}_p \subset L^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow L^2[(\Omega, \mathcal{F}, \mathbb{P}); H]$ is closeable (see Proposition 1.2.1 in [29]). We can define the norm on \mathcal{S}_p :

$$\|F\|_{1,2} := (\mathbb{E}[F^2])^{1/2} + (\mathbb{E}[\|DF\|_H^2])^{1/2},$$

and denote by $\mathbb{D}^{1,2}$ the closure of D on \mathcal{S}_p with respect to this norm. The space $\mathbb{D}^{1,2}$ is then a Hilbert space with inner product

$$\langle F, G \rangle_{1,2} := \mathbb{E}[FG] + \mathbb{E}[\langle DF, DG \rangle_H].$$

2.1. The canonical example. The quintessential example of the above concepts, and the one used in this article, is the case of an m -dimensional Brownian motion W on the canonical basis $(\Omega, \mathcal{F}, \mathbb{P} := (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. Here, $\Omega = C_0([0, T]; \mathbb{R}^m)$, the Banach space of continuous functions $\omega : [0, T] \rightarrow \mathbb{R}^m$ null at zero, \mathbb{P} is Wiener measure, and $\{W_t(\omega) := \omega(t)\}_{t \in [0, T]}$ is m -dimensional Brownian motion with natural filtration \mathbb{F} . The Hilbert space is then $H = L^2([0, T], \mathcal{B}([0, T]), \text{Leb}; \mathbb{R}^m)$, with $\mathcal{B}([0, T])$ the Borel σ -algebra on $[0, T]$ and Leb denoting Lebesgue measure. For brevity we write $H = L^2([0, T]; \mathbb{R}^m)$. An element $h \in H$ is a function $h : [0, T] \rightarrow \mathbb{R}^m$, with $\|h\|_H^2 = \int_0^T \|h(t)\|^2 dt < \infty$ ($\|\cdot\|$ denoting the Euclidean norm), and $\mathbb{W}(h)$ is the Wiener integral

$$\mathbb{W}(h) := \sum_{i=1}^m \int_0^T h_t^i dW_t^i \equiv \int_0^T h_t \cdot dW_t \equiv (h \cdot W)_T.$$

Then $\mathbb{E}[\mathbb{W}(h)] = 0$ and

$$\mathbb{E}[\mathbb{W}(h)\mathbb{W}(g)] = \langle h, g \rangle_H = \int_0^T h_t \cdot g_t dt = \sum_{i=1}^m \int_0^T h_t^i g_t^i dt.$$

For $\varphi \in H = L^2([0, T]; \mathbb{R}^m)$, introduce the *Cameron-Martin* subspace \mathcal{CM} of $\Omega = C_0([0, T]; \mathbb{R}^m)$, consisting of absolutely continuous functions $\Phi : [0, T] \rightarrow \mathbb{R}^m$ with square-integrable derivative φ . That is,

$$(2.2) \quad \Phi_t := \int_0^t \varphi_s ds, \quad \int_0^t \|\varphi_s\|^2 ds < \infty, \quad 0 \leq t \leq T.$$

Transport the Hilbert space structure of H to \mathcal{CM} by defining

$$\langle \Phi, \Psi \rangle_{\mathcal{CM}} := \langle \varphi, \psi \rangle_H = \int_0^T \varphi_t \cdot \psi_t dt, \quad \Psi := \int_0^\cdot \psi_s ds,$$

so \mathcal{CM} is isomorphic to H . Then, for $F \in \mathcal{S}_p$ and $\Phi \in \mathcal{CM}$ as in (2.2), the integration-by-parts formula (2.1) reads as

$$(2.3) \quad \mathbb{E} \left[\int_0^T D_t F \cdot \varphi_t \, dt \right] = \mathbb{E} \left[F \int_0^T \varphi_t \cdot dW_t \right].$$

2.2. The Bismut approach. Bismut [5] developed an alternative formulation of the stochastic calculus of variations, in which the left-hand-side of (2.3) is a directional derivative on Wiener space, and the process φ is any bounded previsible process.

Given a square-integrable functional $F(W)$ of the Brownian paths W , that is, an \mathcal{F}_T -measurable map $F : \Omega \rightarrow \mathbb{R}$ satisfying

$$(2.4) \quad \mathbb{E}[F^2(W)] < \infty,$$

we would like to define a directional derivative in the direction $\Phi \in \Omega$, with $\Phi := \int_0^\cdot \varphi_s \, ds$:

$$\frac{d}{d\epsilon} [F(W + \epsilon\Phi)]|_{\epsilon=0} := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [F(W + \epsilon\Phi) - F(W)].$$

It turns out that one can make sense of the limit, resulting in a version of the integration-by-parts formula (2.3) which holds regardless of whether F is Malliavin differentiable. We formalise this in the following lemma, which encapsulates Bismut's approach to the stochastic calculus of variations.

Lemma 2.1 (Directional derivative on Wiener space). *Let $F \equiv F(W)$ be a square-integrable functional of the Brownian paths W on the Banach space $\Omega = C_0([0, T]; \mathbb{R}^m)$. Let φ be a bounded previsible process, with $\Phi \in \Omega$ defined by $\Phi := \int_0^\cdot \varphi_s \, ds$. Then the map $\epsilon \rightarrow \mathbb{E}[F(W + \epsilon\Phi)]$ is differentiable with*

$$(2.5) \quad \frac{d}{d\epsilon} \mathbb{E}[F(W + \epsilon\Phi)]|_{\epsilon=0} = \mathbb{E}[F(W)(\varphi \cdot W)_T].$$

Remark 2.2. The right-hand-side of (2.5) is $\mathbb{E}[F\mathbb{W}(\varphi)]$, which appears in the integration-by-parts formulae (2.1) and (2.3). Hence, when F is Malliavin-differentiable and $\Phi \in \mathcal{CM} \subset \Omega$, the object in (2.5) is also given by $\mathbb{E}[\langle DF, \varphi \rangle_H]$.

A form of Lemma 2.1 appears in Davis [7] (his Lemma 3) in a one-dimensional set-up, with a functional dependent only on the final value of a diffusion. Below we show how the Bismut approach to Malliavin calculus establishes Lemma 2.1. We give this argument as it is the bedrock of the asymptotic analysis of our control problem. This is a modified version of arguments found in some proofs of the Clark representation formula (see, for instance, the proof of Theorem VIII.2.4 in Revuz and Yor [30]). Our proof is more akin to the proof of Proposition 3.1 in Fournié *et al* [12] (but note there are some typographical errors in that proof).

Proof of Lemma 2.1. We can take φ to be previsible and, in particular, bounded, as we are only concerned with the direction in Ω that φ induces, this being the direction in which we shall compute a directional derivative.

For $\epsilon \in \mathbb{R}$, define the probability measure \mathbb{P}^ϵ by

$$\frac{d\mathbb{P}^\epsilon}{d\mathbb{P}} \Big|_{\mathcal{F}_t} := \mathcal{E}(-\epsilon\varphi \cdot W)_t, \quad 0 \leq t \leq T.$$

By the Girsanov theorem, $W^\epsilon := W + \epsilon\Phi$ is Brownian motion under \mathbb{P}^ϵ , so we have the invariance principle

$$(2.6) \quad \text{Law}(W; \mathbb{P}) = \text{Law}(W^\epsilon; \mathbb{P}^\epsilon),$$

where $\text{Law}(W; \mathbb{P})$ denotes the probability law of W under \mathbb{P} . We shall use (2.6) shortly.

Let \mathbb{E}^ϵ denote expectation under \mathbb{P}^ϵ . Since $\mathbb{P}^\epsilon \sim \mathbb{P}$, we can write

$$(2.7) \quad \mathbb{E}[F(W + \epsilon\Phi)] = \mathbb{E}^\epsilon \left[F(W + \epsilon\Phi) \frac{d\mathbb{P}}{d\mathbb{P}^\epsilon} \right] = \mathbb{E}^\epsilon \left[F(W^\epsilon) \frac{d\mathbb{P}}{d\mathbb{P}^\epsilon} \right].$$

A simple calculation establishes that

$$\frac{d\mathbb{P}}{d\mathbb{P}^\epsilon} = \frac{1}{\mathcal{E}(\epsilon\varphi \cdot W)_T} = \mathcal{E}(\epsilon\varphi \cdot W^\epsilon)_T.$$

Hence, (2.7) becomes

$$\mathbb{E}[F(W + \epsilon\Phi)] = \mathbb{E}^\epsilon [F(W^\epsilon) \mathcal{E}(\epsilon\varphi \cdot W^\epsilon)_T].$$

Using (2.6), we may re-cast the right-hand-side in terms of W and expectation under \mathbb{P} , to obtain

$$(2.8) \quad \mathbb{E}[F(W + \epsilon\Phi)] = \mathbb{E} [F(W) \mathcal{E}(\epsilon\varphi \cdot W)_T].$$

Using this relation we may differentiate $\mathbb{E}[F(W + \epsilon\Phi)]$ with respect to ϵ at $\epsilon = 0$. We have

$$(2.9) \quad \frac{1}{\epsilon} (\mathbb{E}[F(W + \epsilon\Phi) - F(W)]) = \mathbb{E} \left[\frac{1}{\epsilon} (\mathcal{E}(\epsilon\varphi \cdot W)_T - 1) F(W) \right].$$

As we show below,

$$(2.10) \quad \frac{1}{\epsilon} (\mathcal{E}(\epsilon\varphi \cdot W)_t - 1) \rightarrow (\varphi \cdot W)_t, \quad \text{in } L_2, \text{ as } \epsilon \rightarrow 0,$$

for every $t \in [0, T]$. Using this in (2.9) along with the square-integrability condition (2.4) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \left| \frac{1}{\epsilon} (\mathbb{E}[F(W + \epsilon\Phi) - F(W)]) - \mathbb{E}[F(W)(\varphi \cdot W)_T] \right|^2 \\ & \leq C \mathbb{E} \left[\left(\frac{1}{\epsilon} (\mathcal{E}(\epsilon\varphi \cdot W)_T - 1) - (\varphi \cdot W)_T \right)^2 \right], \end{aligned}$$

for some constant C . The expectation converges to zero as $\epsilon \rightarrow 0$, and this establishes the lemma once we prove (2.10).

To establish (2.10), first note that since φ is bounded, $(\epsilon\varphi \cdot W)$ satisfies Novikov's criterion, and therefore $\mathcal{E}(\epsilon\varphi \cdot W)$ is a martingale. Using the representation

$$(2.11) \quad \mathcal{E}(\epsilon\varphi \cdot W)_t = 1 + \epsilon \int_0^t \mathcal{E}(\epsilon\varphi \cdot W)_s \varphi_s \cdot dW_s, \quad 0 \leq t \leq T,$$

the stochastic integral is a martingale and we have

$$(2.12) \quad \mathbb{E} \left[\int_0^t (\mathcal{E}(\epsilon\varphi \cdot W)_s)^2 \|\varphi_s\|^2 ds \right] < \infty, \quad 0 \leq t \leq T.$$

Using (2.11) along with the Itô isometry, we have, for any $t \in [0, T]$,

$$\mathbb{E} \left[\int_0^t (\mathcal{E}(\epsilon\varphi \cdot W)_s - 1)^2 ds \right] = \epsilon^2 \mathbb{E} \left[\int_0^t \int_0^s (\mathcal{E}(\epsilon\varphi \cdot W)_u)^2 \|\varphi_u\|^2 du ds \right].$$

By (2.12), the expectation on the right-hand-side is finite for any value of ϵ . Hence, letting $\epsilon \rightarrow 0$ we have

$$(2.13) \quad \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^t (\mathcal{E}(\epsilon\varphi \cdot W)_s - 1)^2 ds \right] = 0, \quad 0 \leq t \leq T.$$

Now, using (2.11) and the Itô isometry once again, we compute, for any $t \in [0, T]$,

$$\begin{aligned} & \mathbb{E} \left[\left(\frac{1}{\epsilon} (\mathcal{E}(\epsilon \varphi \cdot W)_t - 1) - (\varphi \cdot W)_t \right)^2 \right] \\ &= \mathbb{E} \left[\int_0^t (\mathcal{E}(\epsilon \varphi \cdot W)_s - 1)^2 \|\varphi_s\|^2 ds \right], \end{aligned}$$

which, using (2.13) and the fact that φ is bounded, converges to zero as $\epsilon \rightarrow 0$, and this gives (2.10). \square

Lemma 2.1 implies that

$$(2.14) \quad \mathbb{E}[F(W + \epsilon \Phi) - F(W) - \epsilon F(W)(\varphi \cdot W)_T] \sim O(\epsilon^2).$$

The following lemma is a variant of (2.14). A special case of it appears in Davis [7], in a one-dimensional situation.

Lemma 2.3. *Given the conditions stated in Lemma 2.1, if $\varphi = c\tilde{\varphi}$ for some fixed $\tilde{\varphi}$ and $c \in \mathbb{R}$, then*

$$(2.15) \quad \mathbb{E}[F(W + \epsilon \Phi) - F(W) - \epsilon F(W)(\varphi_t \cdot W)_T] \sim O(c^2 \epsilon^2).$$

Proof. Using (2.8), along with the representation (2.11) with $\varphi = c\tilde{\varphi}$, we have

$$\begin{aligned} & \mathbb{E}[F(W + \epsilon \Phi) - F(W) - \epsilon F(W)(\varphi \cdot W)_T] \\ &= \epsilon^2 c^2 \mathbb{E} \left[F(W) \int_0^T \left(\frac{\mathcal{E}(\epsilon c \tilde{\varphi} \cdot W)_t - 1}{\epsilon c} \right) \tilde{\varphi}_t \cdot dW_t \right]. \end{aligned}$$

Using (2.10) and the square-integrability condition (2.4) as in the proof of Lemma 2.1, it is not hard to see that the expectation is finite as $\epsilon \rightarrow 0$, and this gives (2.15). \square

Of course, for $c = 1$, (2.15) is (2.14).

3. MALLIAVIN ASYMPTOTICS OF A CONTROL PROBLEM

In this section we describe a control problem and analyse it via Malliavin asymptotics. How this type of problem arises in a financial model will be described in subsequent sections.

We have a canonical basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q})$, where \mathbb{Q} represents some ELMM in an incomplete market model, and an m -dimensional \mathbb{Q} -Brownian motion $W^{\mathbb{Q}}$. A state variable $X \in \mathbb{R}^m$ is a functional of the Brownian paths, and follows the Itô process

$$(3.1) \quad dX_t = a_t dt + b_t(dW_t^{\mathbb{Q}} + \epsilon \varphi_t dt).$$

for some m -dimensional adapted process a and $m \times m$ matrix process b , satisfying $\int_0^T \|a_t\| dt < \infty$ and $\int_0^T b_t b_t^* dt < \infty$, where $*$ denotes matrix transposition. In (3.1), ϵ is a small parameter and φ is a control process satisfying $\mathbb{E}^{\mathbb{Q}}[\int_0^T \|\varphi_t\|^2 dt] < \infty$. Denote by $\Phi(\mathbf{M}_f)$ the set of such controls (which, as will be seen in Section 5, corresponds to optimising over martingale measures with finite relative entropy with respect to the physical measure).

It turns out that each choice of the perturbation $\epsilon \varphi$ in (3.1) yields a different martingale measure \mathbb{Q} . Let us write $\mathbb{Q} = \mathbb{Q}(\epsilon)$ to emphasise the dependence on ϵ , so that (3.1) becomes

$$(3.2) \quad dX_t = a_t dt + b_t(dW_t^{\mathbb{Q}(\epsilon)} + \epsilon \varphi_t dt),$$

and in particular, the dynamics of X under $\mathbb{Q}(0)$ are therefore

$$(3.3) \quad dX_t = a_t dt + b_t dW_t^{\mathbb{Q}(0)},$$

where $W^{\mathbb{Q}(0)}$ is a $\mathbb{Q}(0)$ -Brownian motion.

As will be seen in subsequent sections, the state process X is made up of d traded stocks S and $(m - d)$ non-traded factors Y ,

$$X = \begin{pmatrix} S \\ Y \end{pmatrix},$$

such that S is a local $\mathbb{Q}(\epsilon)$ -martingale, for $\epsilon \in \mathbb{R}$. Under these conditions, it turns out that for any integrand θ^ϵ , defined so that the stochastic integral $(\theta^\epsilon \cdot S)$ is a $\mathbb{Q}(\epsilon)$ -martingale, we have the following property, which will be justified in Lemma 5.1.

Property 3.1. *Let θ^ϵ be a d -dimensional adapted process such that $(\theta^\epsilon \cdot S)$ is a $\mathbb{Q}(\epsilon)$ -martingale. Then for all controls φ appearing in (3.2), $(\theta^\epsilon \cdot S)$ and $(\varphi \cdot W^{\mathbb{Q}(\epsilon)})$ are orthogonal $\mathbb{Q}(\epsilon)$ martingales. That is, we have*

$$(3.4) \quad \mathbb{E}^{\mathbb{Q}(\epsilon)}[(\theta^\epsilon \cdot S)_T(\varphi \cdot W^{\mathbb{Q}(\epsilon)})_T] = 0.$$

In particular, this property holds for $\epsilon = 0$.

A square-integrable random variable F is a functional of the paths of the process X . Hence, under $\mathbb{Q}(\epsilon)$, F is a functional of the perturbed Brownian motion $W^{\mathbb{Q}(\epsilon)} + \epsilon\Phi$, with $\Phi := \int_0^\cdot \varphi_s ds$. Write $F^\epsilon \equiv F(W^{\mathbb{Q}(\epsilon)} + \epsilon\Phi)$. Under $\mathbb{Q}(\epsilon)$, F^ϵ will admit a Kunita-Watanabe decomposition of the form

$$(3.5) \quad F^\epsilon = \mathbb{E}^{\mathbb{Q}(\epsilon)}[F^\epsilon] + (\theta^\epsilon \cdot S)_T + (\xi^\epsilon \cdot W^{\mathbb{Q}(\epsilon)})_T,$$

for adapted processes $\theta^\epsilon, \xi^\epsilon$ such that $(\theta^\epsilon \cdot S)$ and $(\xi^\epsilon \cdot W^{\mathbb{Q}(\epsilon)})$ are orthogonal $\mathbb{Q}(\epsilon)$ -martingales. In particular, this will also hold for $\epsilon = 0$.

The control problem we are interested in is of the form

$$(3.6) \quad p := \sup_{\varphi \in \Phi(\mathbf{M}_f)} \mathbb{E}^{\mathbb{Q}(\epsilon)} \left[F^\epsilon - \frac{1}{2} \int_0^T \|\varphi_t\|^2 dt \right],$$

The idea behind the asymptotic expansion is to treat $\epsilon\varphi$ as a perturbation in the dynamics (3.2). We suppose that, for small ϵ , the optimal control $\hat{\varphi}$ will be small. Then, when we expand (3.6) about $\epsilon = 0$, $\mathbb{Q}(0)$ is a measure about which the asymptotic expansion is based. For indifference pricing, it turns out that $\mathbb{Q}(0) = \mathbb{Q}^0 \equiv \mathbb{Q}_E$, the MEMM. For entropy minimisation, we will have $\mathbb{Q}(0) = \mathbb{Q}_M$, the minimal martingale measure. Naturally, for $\epsilon = 0$ the state process loses all dependence on the control φ , so in this case optimal control is zero, and the leading order term will be $\mathbb{E}^{\mathbb{Q}(0)}[F^0]$.

Here is the main result, an asymptotic expansion of the control problem (3.6).

Theorem 3.2. *Let $\epsilon \in \mathbb{R}$ be a small parameter. On the canonical basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{Q}(\epsilon))$, define an m -dimensional Brownian motion $W^{\mathbb{Q}(\epsilon)}$. Let $\Phi_t := \int_0^t \varphi_s ds$ be such that $\mathbb{E}^{\mathbb{Q}(\epsilon)}[\int_0^T \|\varphi_t\|^2 dt] < \infty$, and denote the set of such φ by $\Phi(\mathbf{M}_f)$. Let $F^\epsilon \equiv F(W^{\mathbb{Q}(\epsilon)} + \epsilon\Phi)$ be a square-integrable functional of the perturbed Brownian paths $W^{\mathbb{Q}(\epsilon)} + \epsilon\Phi$, through dependence on the state variable X , which follows (3.2). The control problem with value function (3.6) has asymptotic value given by*

$$p = \mathbb{E}^{\mathbb{Q}(0)}[F^0] + \frac{1}{2} \epsilon^2 \mathbb{E}^{\mathbb{Q}(0)} \left[\int_0^T \|\xi_t^0\|^2 dt \right] + O(\epsilon^4),$$

where ξ^0 is the integrand in the Kunita-Watanabe decomposition (3.5) of F^0 for $\epsilon = 0$.

Proof. The idea of the proof is to consider a control problem where we fix a reference measure and perturb the state process, as opposed to fixing the state variable and modulating the measure. To this end, fix a canonical space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{M})$ with reference probability measure \mathbb{M} and an \mathbb{M} -Brownian motion W , and consider the perturbed process

$$dX_t^\epsilon = a_t^\epsilon dt + b_t^\epsilon (dW_t + \epsilon \varphi_t dt).$$

Here, we write X^ϵ to emphasise the dependence on ϵ . The parameters a^ϵ, b^ϵ are adapted functionals of the paths of X^ϵ , and hence of the Brownian paths. For $\epsilon = 0$, we have a process X^0 following

$$dX_t^0 = a_t^0 dt + b_t^0 dW_t.$$

The control problem is then written as one over the fixed measure \mathbb{M} , to maximise over φ the objective function

$$\mathbb{E}^{\mathbb{M}} \left[F^\epsilon - \frac{1}{2} \int_0^T \|\varphi_t\|^2 dt \right].$$

where, as before, F^ϵ denotes a functional of X^ϵ , for $\epsilon \in \mathbb{R}$.

We now use Lemma 2.1 to differentiate $\mathbb{E}^{\mathbb{M}}[F^\epsilon]$ with respect to ϵ at $\epsilon = 0$. Then the objective function of the control problem is approximated as

$$(3.7) \quad \mathbb{E}^{\mathbb{M}} \left[F^\epsilon - \frac{1}{2} \int_0^T \|\varphi_t\|^2 dt \right] = \mathbb{E}^{\mathbb{M}} \left[F^0 + \epsilon F^0(\varphi \cdot W)_T - \frac{1}{2} \int_0^T \|\varphi_t\|^2 dt \right] + o(\epsilon),$$

where $o(\epsilon)$ denotes terms of smaller order than linear in ϵ .

One now invokes the Kunita-Watanabe decomposition of F^0 under \mathbb{M} , which in the original set-up corresponds to taking $\epsilon = 0$ in (3.5), so in this context reads as

$$(3.8) \quad F^0 = \mathbb{E}^{\mathbb{M}}[F^0] + (\theta^0 \cdot S)_T + (\xi^0 \cdot W)_T,$$

for adapted processes θ^0, ξ^0 such that $(\theta^0 \cdot S)$ and $(\xi^0 \cdot W)$ are orthogonal martingales. We also have the analogue of (3.4) with $\epsilon = 0$, which in this context says that for all controls φ , $(\theta^0 \cdot S)$ and $(\varphi \cdot W)$ are orthogonal \mathbb{M} martingales, so we have

$$(3.9) \quad \mathbb{E}^{\mathbb{M}}[(\theta^0 \cdot S)_T(\varphi \cdot W)_T] = 0.$$

Using the Kunita-Watanabe decomposition (3.8) we substitute for F^0 in the linear (in ϵ) term in (3.7), and use (3.9) and the martingale property of $(\varphi \cdot W)$. This gives

$$\mathbb{E}^{\mathbb{M}} \left[F^\epsilon - \frac{1}{2} \int_0^T \|\varphi_t\|^2 dt \right] = \mathbb{E}^{\mathbb{M}} \left[F^0 + \int_0^T \left(\epsilon \xi_t^0 \cdot \varphi_t - \frac{1}{2} \|\varphi_t\|^2 \right) dt \right] + o(\epsilon).$$

We maximise the right-hand-side over φ by choosing $\varphi = \epsilon \xi^0$. This yields that the control problem has asymptotic value given by

$$\mathbb{E}^{\mathbb{M}}[F^0] + \frac{1}{2} \epsilon^2 \mathbb{E}^{\mathbb{M}} \left[\int_0^T \|\xi_t^0\|^2 dt \right] + O(\epsilon^4),$$

with the correction term being of order ϵ^4 due to Lemma 2.3. This is the result we want, since the dynamics of X^0 under \mathbb{M} match the $\mathbb{Q}(0)$ -dynamics of the original state process X in (3.3). □

4. DYNAMIC DUAL REPRESENTATIONS OF INDIFFERENCE PRICE PROCESSES

In this section we derive a dynamic dual stochastic control representation for the exponential indifference price process of a European claim in a locally bounded semi-martingale market. This will form the basis for our asymptotic expansion of the indifference price. Our representation is a slight deviation from the usual way of expressing the indifference price in terms of relative entropy. Although the material in this section is mainly classical, we want a unified treatment that gives dynamic results for unbounded claims, and this is not readily available in one compact account.

Our approach is to begin with the seminal representation of Grandits and Rheinländer [14] and Kabanov and Stricker [18] for an entropy-minimising measure, to establish a dynamic version of this (Corollary 4.4), and to use this to establish a dynamic version (Theorem 4.5) of the duality result of Delbaen *et al* [8]. This result has been obtained for a bounded claim by Mania and Schweizer [23]. We carry out this program for a claim satisfying exponential moment conditions akin to those in Becherer [2]. Once we establish duality for the investment problem with random endowment, we obtain a dynamic version of the classical dual indifference price representation (Corollary 4.6). Then we derive a dynamic result on the entropic distance between measures (Proposition 4.7) using the results of [14, 18] once more, and this allows us to convert the classical indifference price representation to our required representation in Lemma 4.8.

The setting is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right-continuity and completeness, where $T \in (0, \infty)$ is a fixed time horizon. We assume that \mathcal{F}_0 is trivial and that $\mathcal{F} = \mathcal{F}_T$.

The discounted prices of d stocks are modelled by a positive locally bounded semi-martingale S . Since we work with discounted assets, our formulae are unencumbered by any interest rate adjustments. The class \mathbf{M} of equivalent local martingale measures (ELMMs) \mathbb{Q} is of course defined by

$$\mathbf{M} := \{\mathbb{Q} \sim \mathbb{P} | S \text{ is a } \mathbb{Q}\text{-local martingale}\},$$

and is assumed non-empty. This assumption is a classical one, consistent with the absence of arbitrage opportunities, in accordance with Delbaen and Schachermayer [9].

Denote by $Z^{\mathbb{Q}}$ the density process with respect to \mathbb{P} of any $\mathbb{Q} \in \mathbf{M}$. We write $Z^{\mathbb{Q}, \mathbb{M}}$ for the density process of $\mathbb{Q} \in \mathbf{M}$ with respect to any measure \mathbb{M} other than the physical measure \mathbb{P} .

For $0 \leq t \leq T$, we write $Z_{t,T}^{\mathbb{Q}} := Z_T^{\mathbb{Q}}/Z_t^{\mathbb{Q}}$, with a similar convention for any positive process. The *conditional relative entropy* between $\mathbb{Q} \in \mathbf{M}$ and \mathbb{P} is the process defined by

$$(4.1) \quad I_t(\mathbb{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}}[\log Z_{t,T}^{\mathbb{Q}} | \mathcal{F}_t], \quad 0 \leq t \leq T,$$

provided this is almost surely finite. Define the subset of \mathbf{M} given by

$$\mathbf{M}_f := \{\mathbb{Q} \in \mathbf{M} | I_0(\mathbb{Q}|\mathbb{P}) < \infty\},$$

and we assume throughout that this set of ELMMs with finite relative entropy is non-empty: $\mathbf{M}_f \neq \emptyset$. By Theorem 2.1 of Frittelli [13], this implies that there exists a unique $\mathbb{Q}^0 \in \mathbf{M}_f$, the minimal entropy martingale measure (MEMM), that minimises $I_0(\mathbb{Q}|\mathbb{P})$ over all $\mathbb{Q} \in \mathbf{M}_f$. It is well-known (for example, Proposition 4.1 of Kabanov and Stricker [18]) that the density process $Z^{\mathbb{Q}^0}$ also minimises the conditional relative entropy process $I(\mathbb{Q}|\mathbb{P})$ between $\mathbb{Q} \in \mathbf{M}_f$ and \mathbb{P} .

The density process of one martingale measure with respect to another is simply the ratio of their density processes with respect to \mathbb{P} , as shown in the following lemma.

Lemma 4.1. *Let $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathbf{M}_f$ have density processes $Z^{\mathbb{Q}_1}, Z^{\mathbb{Q}_2}$ with respect to \mathbb{P} . Then the density process of \mathbb{Q}_1 with respect to \mathbb{Q}_2 is $Z^{\mathbb{Q}_1}/Z^{\mathbb{Q}_2}$.*

Proof. Denote by $Z^{\mathbb{Q}_1, \mathbb{Q}_2}$ the density process of \mathbb{Q}_1 with respect to \mathbb{Q}_2 . We have

$$Z_T^{\mathbb{Q}_1, \mathbb{Q}_2} := \frac{d\mathbb{Q}_1}{d\mathbb{Q}_2} = \frac{d\mathbb{Q}_1}{d\mathbb{P}} \left(\frac{d\mathbb{Q}_2}{d\mathbb{P}} \right)^{-1} = \frac{Z_T^{\mathbb{Q}_1}}{Z_T^{\mathbb{Q}_2}}.$$

Hence, the \mathbb{Q}_2 -martingale $Z^{\mathbb{Q}_1, \mathbb{Q}_2}$ is given by

$$\begin{aligned} Z_t^{\mathbb{Q}_1, \mathbb{Q}_2} = \mathbb{E}^{\mathbb{Q}_2}[Z_T^{\mathbb{Q}_1, \mathbb{Q}_2} | \mathcal{F}_t] &= \mathbb{E}^{\mathbb{Q}_2} \left[\frac{Z_T^{\mathbb{Q}_1}}{Z_T^{\mathbb{Q}_2}} \middle| \mathcal{F}_t \right] \\ &= \frac{1}{Z_t^{\mathbb{Q}_2}} \mathbb{E}[Z_T^{\mathbb{Q}_1} | \mathcal{F}_t] = \frac{Z_t^{\mathbb{Q}_1}}{Z_t^{\mathbb{Q}_2}}, \quad 0 \leq t \leq T, \end{aligned}$$

the penultimate equality following from the Bayes rule applied between \mathbb{Q}_2 and \mathbb{P} , and the final equality from the fact that $Z^{\mathbb{Q}_1}$ is a \mathbb{P} -martingale. \square

A financial agent trades S and has risk preferences represented by the exponential utility function

$$U(x) = -\exp(-\alpha x), \quad \alpha > 0, \quad x \in \mathbb{R},$$

with risk aversion coefficient α . A European contingent claim has \mathcal{F}_T -measurable payoff F . Following Becherer [2] and others, we assume that F satisfies suitable exponential moment conditions:

$$(4.2) \quad \mathbb{E}[\exp((\alpha + \epsilon)F)] < \infty, \quad \mathbb{E}[\exp(-\epsilon F)] < \infty, \quad \text{for some } \epsilon > 0.$$

Condition (4.2) is sufficient to guarantee that F is \mathbb{Q} -integrable for any $\mathbb{Q} \in \mathbf{M}_f$ (see for example Lemma A.1 in Becherer [2]).

4.1. The dynamic primal and dual problems. The set Θ of admissible trading strategies is defined as the set of S -integrable processes θ such that the stochastic integral $\theta \cdot S$ is a \mathbb{Q} -martingale for every $\mathbb{Q} \in \mathbf{M}_f$, where θ is a d -dimensional vector representing the number of shares of each stock in the vector S . It is well-known [2, 8, 18, 31, 32] that there are a number of possible choices for a feasible set of permitted strategies, which all lead to the same value for the dual problem, defined further below, and it is on this latter problem that our analysis will be centred on. For any $t \in [0, T]$, fix an \mathcal{F}_t -measurable random variable x_t , representing initial capital. Let Θ_t denote admissible strategies beginning at t .

The primal problem is to maximise expected utility of terminal wealth generated from trading S and paying the claim payoff at T . The maximal expected utility process is

$$(4.3) \quad u_t^F(x_t) := \operatorname{ess\,sup}_{\theta \in \Theta_t} \mathbb{E} \left[-e^{-\alpha(x_t + \int_t^T \theta_u \cdot dS_u - F)} \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T,$$

with $\int_t^T \theta_u \cdot dS_u = \sum_{i=1}^d \int_t^T \theta_u^i dS_u^i$.

We shall use the notational convention whereby setting $F = 0$ in (4.3) signifies the corresponding quantity in the problem without the claim. Hence, the classical investment problem without the claim has maximal expected utility process u^0 . Denote the optimiser in (4.3) by θ^F , so θ^0 is the optimiser in the problem without the claim.

The utility indifference price process for the claim, $p(\alpha)$, is defined by

$$(4.4) \quad u_t^F(x_t + p_t(\alpha)) = u_t^0(x_t), \quad 0 \leq t \leq T.$$

It is well-known that $p(\alpha)$ has no dependence on the starting capital. The hedging strategy associated with this pricing mechanism is $\theta(\alpha)$, defined by

$$(4.5) \quad \theta(\alpha) := \theta^F - \theta^0.$$

The dual problem to (4.3) is defined by

$$(4.6) \quad I_t^F := \operatorname{ess\,inf}_{\mathbb{Q} \in \mathbf{M}_f} \left[I_t(\mathbb{Q}|\mathbb{P}) - \alpha \mathbb{E}^{\mathbb{Q}}[F|\mathcal{F}_t] \right], \quad 0 \leq t \leq T.$$

Denote the optimiser in (4.6) by \mathbb{Q}^F , so the optimiser without the claim is \mathbb{Q}^0 , the MEMM.

It is well-known (at least in a static context) that if we define the measure $\mathbb{P}_F \sim \mathbb{P}$ by

$$(4.7) \quad \frac{d\mathbb{P}_F}{d\mathbb{P}} := \frac{\exp(\alpha F)}{\mathbb{E}[\exp(\alpha F)]},$$

then we can use \mathbb{P}_F instead of \mathbb{P} as our reference measure, and this removes the claim from the primal and dual problems. In the dual picture, therefore, \mathbb{Q}^F is the martingale measure which minimises the relative entropy between any $\mathbb{Q} \in \mathbf{M}_f$ and \mathbb{P}_F . These properties of \mathbb{P}_F are well-known in a static context from Delbaen *et al* [8]. The dynamic analogue of these arguments is given below.

Note that if we use \mathbb{P}_F instead of \mathbb{P} as reference measure, one could (in principle) define a set $\mathbf{M}_f(\mathbb{P}_F)$ of ELMMs with finite relative entropy with respect to \mathbb{P}_F , but it is well-known that $\mathbf{M}_f(\mathbb{P}_F) = \mathbf{M}_f(\mathbb{P})$ (see the statement and proof of Lemma A.1 in Becherer [2], for example) so we simply write \mathbf{M}_f .

Define the \mathbb{P} -martingale M^F as the density process of \mathbb{P}_F with respect to \mathbb{P} :

$$M_t^F := \frac{d\mathbb{P}_F}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \mathbb{E} \left[\frac{d\mathbb{P}_F}{d\mathbb{P}} \Big| \mathcal{F}_t \right] = \frac{\mathbb{E}[e^{\alpha F} | \mathcal{F}_t]}{\mathbb{E}[e^{\alpha F}]}, \quad 0 \leq t \leq T,$$

which satisfies, for any integrable \mathcal{F}_T -measurable random variable V ,

$$(4.8) \quad \mathbb{E}^{\mathbb{P}_F}[V | \mathcal{F}_t] = \frac{1}{M_t^F} \mathbb{E}[M_t^F V | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

We “remove the claim” from the primal problem using the measure \mathbb{P}_F as follows. Using (4.8) we convert (4.3) to

$$u_t^F(x_t) := \mathbb{E}[e^{\alpha F} | \mathcal{F}_t] \operatorname{ess\,sup}_{\theta \in \Theta_t} \mathbb{E}^{\mathbb{P}_F} \left[-e^{-\alpha(x_t + \int_t^T \theta_u \cdot dS_u)} \Big| \mathcal{F}_t \right], \quad 0 \leq t \leq T,$$

from which it is apparent that one may optimise over strategies in a problem without the claim and with \mathbb{P}_F as reference measure. The same approach also works, of course, for the dual problem, as we show below. We shall need the following simple result relating the density process of any $\mathbb{Q} \in \mathbf{M}_f$ with respect to \mathbb{P} to its counterpart with respect to \mathbb{P}_F .

Lemma 4.2. *For any $\mathbb{Q} \in \mathbf{M}_f$, the density processes $Z^{\mathbb{Q}}$ and $Z^{\mathbb{Q}, \mathbb{P}_F}$ are related by*

$$Z_t^{\mathbb{Q}} = M_t^F Z_t^{\mathbb{Q}, \mathbb{P}_F}, \quad 0 \leq t \leq T.$$

Proof. For $\mathbb{Q} \in \mathbf{M}_f$, we have

$$\begin{aligned} Z_t^{\mathbb{Q}, \mathbb{P}_F} &= \mathbb{E}^{\mathbb{P}_F} \left[\frac{d\mathbb{Q}}{d\mathbb{P}_F} \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}_F} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle/ \frac{d\mathbb{P}_F}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{P}_F} \left[\frac{1}{M_t^F} \frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \\ &= \frac{1}{M_t^F} \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \\ &= \frac{Z_t^{\mathbb{Q}}}{M_t^F}, \quad 0 \leq t \leq T, \end{aligned}$$

where we have used (4.8). □

Applying Lemma 4.2 in turn at $t \leq T$ and at T , we obtain

$$(4.9) \quad Z_{t,T}^{\mathbb{Q}, \mathbb{P}_F} = \frac{Z_{t,T}^{\mathbb{Q}}}{M_{t,T}^F} = \frac{\mathbb{E}[e^{\alpha F} | \mathcal{F}_t]}{e^{\alpha F}} Z_{t,T}^{\mathbb{Q}}, \quad 0 \leq t \leq T.$$

We use this to “remove the claim” from the dual problem (4.6): compute, for any $\mathbb{Q} \in \mathbf{M}_f$,

$$\begin{aligned} I_t(\mathbb{Q}^F | \mathbb{P}_F) &= \mathbb{E}^{\mathbb{Q}}[\log Z_{t,T}^{\mathbb{Q}, \mathbb{P}_F} | \mathcal{F}_t] \\ &= I_t(\mathbb{Q} | \mathbb{P}) - \alpha \mathbb{E}^{\mathbb{Q}}[F | \mathcal{F}_t] + \log(\mathbb{E}[e^{\alpha F} | \mathcal{F}_t]), \quad 0 \leq t \leq T. \end{aligned}$$

Using this in (4.6), we obtain

$$(4.10) \quad I_t^F = \operatorname{ess\,inf}_{\mathbb{Q} \in \mathbf{M}_f} [I_t(\mathbb{Q} | \mathbb{P}_F)] - \log(\mathbb{E}[e^{\alpha F} | \mathcal{F}_t]), \quad 0 \leq t \leq T.$$

Since the last term on the right-hand-side does not depend on \mathbb{Q} , we see that we can reduce the dual problem to the problem

$$I_t(\mathbb{Q} | \mathbb{P}_F) \longrightarrow \min!,$$

so that \mathbb{Q}^F minimises $I(\mathbb{Q} | \mathbb{P}_F)$, and, when $F = 0$, \mathbb{Q}^0 is the MEMM.

4.2. The fundamental duality. The duality results we need follow from the representation below for $Z^{\mathbb{Q}^F, \mathbb{P}_F}$, originally proven independently (to the best of our knowledge) by Grandits and Rheinländer [14] and Kabanov and Stricker [18] for $F = 0$ (and hence for $Z^{\mathbb{Q}^0}$), but which applies equally well to \mathbb{Q}^F if we use \mathbb{P}_F as reference measure. Both [14] and [18] prove the result for a market involving a locally bounded semi-martingale S . This has been generalised to a general semi-martingale by Biagini and Frittelli [4].

Property 4.3 ([14, 18]). *The density of the dual minimiser \mathbb{Q}^F in (4.6) with respect to the measure \mathbb{P}_F defined in (4.7) is given by*

$$(4.11) \quad \frac{d\mathbb{Q}^F}{d\mathbb{P}_F} \equiv Z_T^{\mathbb{Q}^F, \mathbb{P}_F} = c_F \exp(-\alpha(\theta^F \cdot S)_T), \quad c_F \in \mathbb{R}_+,$$

where $\theta^F \in \Theta$ is the optimal strategy in the primal problem (4.3) and the stochastic integral $(\theta^F \cdot S)$ is a \mathbb{Q} -martingale for any $\mathbb{Q} \in \mathbf{M}_f$.

We convert this to the dynamic result below, in which we also restore \mathbb{P} as reference measure.

Corollary 4.4. *The density process $Z^{\mathbb{Q}^F}$ of the dual minimiser \mathbb{Q}^F in (4.6) satisfies, for $t \in [0, T]$,*

$$(4.12) \quad Z_{t,T}^{\mathbb{Q}^F} = \exp \left[I_t(\mathbb{Q}^F | \mathbb{P}) - \alpha \left(\mathbb{E}^{\mathbb{Q}^F} [F | \mathcal{F}_t] + \int_t^T \theta_u^F \cdot dS_u - F \right) \right],$$

where $\theta^F \in \Theta$ is the optimal strategy in the primal problem (4.3).

Proof. First, we obtain a dynamic version of (4.11). Using (4.11) and the \mathbb{Q}^F -martingale property of $(\theta^F \cdot S)$, we have

$$\begin{aligned} I_t(\mathbb{Q}^F | \mathbb{P}_F) &= \mathbb{E}^{\mathbb{Q}^F} [\log Z_{t,T}^{\mathbb{Q}^F, \mathbb{P}_F} | \mathcal{F}_t] \\ &= \mathbb{E}^{\mathbb{Q}^F} [\log c_F - \alpha(\theta^F \cdot S)_T | \mathcal{F}_t] - \log Z_t^{\mathbb{Q}^F, \mathbb{P}_F} \\ &= \log c_F - \alpha(\theta^F \cdot S)_t - \log Z_t^{\mathbb{Q}^F, \mathbb{P}_F}, \quad 0 \leq t \leq T. \end{aligned}$$

Using this in turn at $t \leq T$ and T we obtain

$$Z_{t,T}^{\mathbb{Q}^F, \mathbb{P}_F} = c_t^F \exp \left(-\alpha \int_t^T \theta_u^F \cdot dS_u \right), \quad c_t^F := \exp(I_t(\mathbb{Q}^F | \mathbb{P}_F)), \quad 0 \leq t \leq T,$$

which is a dynamic version of (4.11). Using this along with (4.9) and (4.10) we obtain

$$Z_{t,T}^{\mathbb{Q}^F} = \exp \left(I_t^F - \alpha \int_t^T \theta_u^F \cdot dS_u + \alpha F \right), \quad 0 \leq t \leq T.$$

Finally, using the definition (4.6) of I^F gives the result. \square

Corollary 4.4 is nothing more than a dynamic version of the classical result of Grandits and Rheinländer [14] and Kabanov and Stricker [18] for the MEMM, with the added generalisation of allowing for \mathbb{P}_F as reference measure. It leads immediately to the duality result below, a dynamic version of the duality in Delbaen *et al* [8]. This result is stated in Mania and Schweizer [23] for a bounded claim. We give a proof to highlight that the boundedness condition on the claim is not needed.

Theorem 4.5 ([8, 2, 18, 23]). *Suppose the claim payoff F satisfies the exponential moment conditions (4.2). Then the maximal expected utility process in (4.3) and the optimal dual process in (4.6) are related by*

$$(4.13) \quad u_t^F(x_t) = -\exp(-\alpha x_t - I_t^F), \quad 0 \leq t \leq T.$$

Proof. We compute the primal optimal expected utility process and use Corollary 4.4 to substitute for the stochastic integral $(\theta^F \cdot S)$:

$$\begin{aligned} u_t^F(x_t) &= \mathbb{E} \left[-e^{-\alpha(x_t + \int_t^T \theta_u^F \cdot dS_u - F)} \middle| \mathcal{F}_t \right] \\ &= -e^{-\alpha x_t} \mathbb{E} [Z_{t,T}^{\mathbb{Q}^F} \exp(-I_t^F) | \mathcal{F}_t] \quad (\text{using Corollary 4.4}) \\ &= -\exp(-\alpha x_t - I_t^F), \quad 0 \leq t \leq T. \end{aligned}$$

\square

Using this theorem and the definition of the indifference price we obtain the following dual representation of the indifference price process, a dynamic version of the classical representation.

Corollary 4.6. *The indifference price process has the dual representation*

$$(4.14) \quad p_t(\alpha) = -\frac{1}{\alpha}(I_t^F - I_t^0), \quad 0 \leq t \leq T.$$

Written out explicitly, (4.14) can be re-cast into the more familiar form

$$(4.15) \quad p_t(\alpha) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathbf{M}_f} \left[\mathbb{E}^{\mathbb{Q}}[F | \mathcal{F}_t] - \frac{1}{\alpha} (I_t(\mathbb{Q} | \mathbb{P}) - I_t(\mathbb{Q}^0 | \mathbb{P})) \right], \quad 0 \leq t \leq T.$$

The two conditional entropy terms in (4.15) can in fact be condensed into one, using the following proposition.

Proposition 4.7. *The conditional entropy process I satisfies the property that, for any equivalent local martingale measure $\mathbb{Q} \in \mathbf{M}_f$,*

$$(4.16) \quad I_t(\mathbb{Q} | \mathbb{P}) - I_t(\mathbb{Q}^0 | \mathbb{P}) = I_t(\mathbb{Q} | \mathbb{Q}^0), \quad 0 \leq t \leq T.$$

Proof. For any $\mathbb{Q} \in \mathbf{M}_f$, the conditional entropy process $I(\mathbb{Q} | \mathbb{Q}^0)$ is given by

$$(4.17) \quad \begin{aligned} I_t(\mathbb{Q} | \mathbb{Q}^0) &:= \mathbb{E}^{\mathbb{Q}}[\log Z_{t,T}^{\mathbb{Q}, \mathbb{Q}^0} | \mathcal{F}_t] \\ &= \mathbb{E}^{\mathbb{Q}}[\log Z_{t,T}^{\mathbb{Q}} - \log Z_{t,T}^{\mathbb{Q}^0} | \mathcal{F}_t] \\ &= I_t(\mathbb{Q} | \mathbb{P}) - \mathbb{E}^{\mathbb{Q}}[\log Z_{t,T}^{\mathbb{Q}^0} | \mathcal{F}_t], \quad 0 \leq t \leq T. \end{aligned}$$

We have the dynamic version of the Grandits-Rheinländer [14] representation of the MEMM, given by (4.12) for $F = 0$:

$$Z_{t,T}^{\mathbb{Q}^0} = \exp \left(I_t(\mathbb{Q}^0 | \mathbb{P}) - \alpha \int_t^T \theta_u^0 \cdot dS_u \right), \quad 0 \leq t \leq T,$$

where the optimal investment strategy $\theta^0 \in \Theta$, so $(\theta^0 \cdot S)$ is a \mathbb{Q} -martingale, for any $\mathbb{Q} \in \mathbf{M}_f$. Using this in (4.17) we obtain (4.16). \square

Using Proposition 4.7 in the classical dual stochastic control representation (4.14) of the indifference price process, we immediately obtain the following form for $p(\alpha)$, which will form the basis for our asymptotic expansion of the indifference price process.

Lemma 4.8. *The indifference price process is given by the dual stochastic control representation*

$$p_t(\alpha) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathbf{M}_f} \left[\mathbb{E}^{\mathbb{Q}}[F | \mathcal{F}_t] - \frac{1}{\alpha} I_t(\mathbb{Q} | \mathbb{Q}^0) \right], \quad 0 \leq t \leq T.$$

Proof. Use (4.16) in (4.14). \square

Remark 4.9. A version of Lemma 4.8 for American claims was given in Leung *et al* [22] in a stochastic volatility scenario (see their Theorem 8).

Remark 4.10. The optimiser in Lemma 4.8 is also the optimiser in (4.15), that is, \mathbb{Q}^F .

5. INDIFFERENCE VALUATION IN AN INCOMPLETE ITÔ PROCESS MARKET

In this section we apply the indifference pricing formula from Lemma 4.8 in an Itô process setting, and we show how it leads to a control problem of the form in Section 3.

We have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the standard augmented filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ associated with an m -dimensional Brownian motion W . On this space we have a financial market with (for simplicity) zero interest rate. The price processes of $d < m$ stocks are given by the vector $S = (S^1, \dots, S^d)^*$, where $S = (S_t)_{0 \leq t \leq T}$ follows the Itô process

$$(5.1) \quad dS_t = \text{diag}_d(S_t)[\mu_t^S dt + \sigma_t dW_t],$$

with $\text{diag}_d(\cdot)$ denoting the $(d \times d)$ matrix with zero entries off the main diagonal. The d -dimensional appreciation rate vector μ^S and the $(d \times m)$ volatility matrix σ are \mathbb{F} -progressively measurable processes, satisfying $\int_0^T \|\mu_t^S\| dt < \infty$ and $\int_0^T \sigma_t \sigma_t^* dt < \infty$, almost surely. The volatility matrix σ_t has full rank for every $t \in [0, T]$, so that the matrix $(\sigma_t \sigma_t^*)^{-1}$ is well-defined, as is the m -dimensional relative risk process given by

$$(5.2) \quad \lambda_t := \sigma_t^* (\sigma_t \sigma_t^*)^{-1} \mu_t^S, \quad 0 \leq t \leq T.$$

For $d < m$, this market is incomplete. We also have a vector $Y = (Y^1, \dots, Y^{m-d})^*$ of $(m - d)$ non-traded factors. These could be the prices of non-traded assets, or of factors such as stochastic volatilities and correlations. This framework is general enough to encompass multi-dimensional versions of basis risk models as well as multi-factor stochastic volatility models, with no Markovian structure needed. We assume that Y follows the Itô process

$$dY_t = \text{diag}_{m-d}(Y_t)[\mu_t^Y dt + \beta_t dW_t],$$

for an $(m - d)$ -dimensional progressively measurable vector μ^Y satisfying $\int_0^T \|\mu_t^Y\| dt < \infty$, almost surely, and an $(m - d) \times m$ -dimensional progressively measurable matrix β satisfying $\int_0^T \beta_t \beta_t^* dt < \infty$, almost surely.

A European contingent claim has \mathcal{F}_T -measurable payoff F depending on the evolution of (S, Y) . We assume $F \in L_2(\mathbb{Q})$, for any ELMM $\mathbb{Q} \in \mathbf{M}_f$.

Measures $\mathbb{Q} \sim \mathbb{P}$ have density processes with respect to \mathbb{P} of the form

$$(5.3) \quad Z_t^{\mathbb{Q}} = \mathcal{E}(-q \cdot W)_t, \quad 0 \leq t \leq T,$$

for some m -dimensional process q satisfying $\int_0^T \|q_t\|^2 dt < \infty$ almost surely. If $Z^{\mathbb{Q}}$ is the density of an equivalent local martingale measure, then it is a \mathbb{P} -martingale, and q satisfies

$$(5.4) \quad \mu_t^S - \sigma_t^S q_t = \mathbf{0}_d, \quad 0 \leq t \leq T,$$

where $\mathbf{0}_d$ denotes the d -dimensional zero vector. As the market is incomplete, there will be an infinite number of solutions q to the equations (5.4), and the ELMMs \mathbb{Q} are in one-to-one correspondence with processes q satisfying (5.4) and such that $\mathcal{E}(-q \cdot W)$ is a \mathbb{P} -martingale. Denote by $\mathbf{Q}(\mathbf{M}_f)$ the set of integrands q in (5.3) which correspond to $\mathbb{Q} \in \mathbf{M}_f$.

By the Girsanov theorem, the process $W^{\mathbb{Q}}$ defined by

$$(5.5) \quad W_t^{\mathbb{Q}} := W_t + \int_0^t q_u du, \quad 0 \leq t \leq T,$$

is an m -dimensional \mathbb{Q} -Brownian motion. The dynamics of the stocks and non-traded factors under \mathbb{Q} are then

$$(5.6) \quad dS_t = \text{diag}_d(S_t) \sigma_t^S dW_t^{\mathbb{Q}},$$

$$(5.7) \quad dY_t = \text{diag}_{m-d}(Y_t)[(\mu_t^Y - \beta_t q_t) dt + \beta_t dW_t^{\mathbb{Q}}].$$

If we choose $q = \lambda$, given by (5.2), we obtain the minimal martingale measure \mathbb{Q}_M , while the density process of the MEMM \mathbb{Q}^0 is $Z^{\mathbb{Q}^0} = \mathcal{E}(-q^0 \cdot W)$, for some integrand q^0 .

Denote by $H^2(\mathbb{Q})$ the space of L^2 -bounded continuous \mathbb{Q} -martingales M (that is, $\sup_{t \in [0, T]} \mathbb{E}^{\mathbb{Q}}[M_t^2] < \infty$), and by $H_0^2(\mathbb{Q})$ the subset of elements of $H^2(\mathbb{Q})$ null at zero. By Proposition IV.1.23 and Corollary IV.1.25 in Revuz and Yor [30], $H^2(\mathbb{Q})$ is also the space of martingales M such that $\mathbb{E}^{\mathbb{Q}}[[M]_T] < \infty$.

Denoting $\Lambda^{\mathbb{Q}} := (q \cdot W^{\mathbb{Q}})$, then using (5.3) and (5.5), $\log Z^{\mathbb{Q}} = -\Lambda^{\mathbb{Q}} + [\Lambda^{\mathbb{Q}}]/2$, so the relative entropy between $\mathbb{Q} \in \mathbf{M}_f$ and \mathbb{P} is given by

$$0 \leq I_0(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}} \left[-\Lambda_T^{\mathbb{Q}} + \frac{1}{2}[\Lambda^{\mathbb{Q}}]_T \right] < \infty,$$

the last inequality true by assumption. The finiteness and non-negativity of this relative entropy yields that both expectations above are finite. Precisely, we have $\mathbb{E}^{\mathbb{Q}}[\Lambda_T^{\mathbb{Q}}] > -\infty$ and, in particular, $\mathbb{E}^{\mathbb{Q}}[[\Lambda^{\mathbb{Q}}]_T] < \infty$, the latter condition implying that $\Lambda^{\mathbb{Q}} \in H^2(\mathbb{Q})$. Therefore,

$$(5.8) \quad \Lambda^{\mathbb{Q}} := (q \cdot W^{\mathbb{Q}}) \text{ is a } \mathbb{Q}\text{-martingale, for all } \mathbb{Q} \in \mathbf{M}_f.$$

This will be useful in computing the conditional relative entropy $I(\mathbb{Q}|\mathbb{Q}^0)$.

Using (5.5) in turn for \mathbb{Q} and \mathbb{Q}^0 , we have

$$(5.9) \quad W_t^{\mathbb{Q}} = W_t^{\mathbb{Q}^0} + \int_0^t (q_t - q_t^0) dt, \quad 0 \leq t \leq T,$$

where $W^{\mathbb{Q}^0}$ is a \mathbb{Q}^0 -Brownian motion.

Note that since both q and q^0 satisfy (5.4), we have

$$(5.10) \quad \sigma_t(q_t - q_t^0) = \mathbf{0}_d, \quad 0 \leq t \leq T,$$

which we shall use later.

Using (5.9), we can write the \mathbb{Q} -dynamics of Y in (5.7) as

$$dY_t = \text{diag}_{m-d}(Y_t)[(\mu_t^Y - \beta_t q_t^0) dt + \beta_t (dW_t^{\mathbb{Q}} - (q_t - q_t^0) dt)].$$

The point of this representation is that the \mathbb{Q} -dynamics of Y may be interpreted as a perturbation of the \mathbb{Q}^0 -dynamics, since setting $q = q^0$ gives the dynamics under the MEMM \mathbb{Q}^0 , with the Brownian motion $W^{\mathbb{Q}}$ also being modulated by the choice of q .

Using (5.5) and (5.9), the density process of \mathbb{Q} with respect to \mathbb{Q}^0 is

$$Z_t^{\mathbb{Q}, \mathbb{Q}^0} = \frac{Z_t^{\mathbb{Q}}}{Z_t^{\mathbb{Q}^0}} = \frac{\mathcal{E}(-q \cdot W)_t}{\mathcal{E}(-q^0 \cdot W)_t} = \mathcal{E}(-(q - q^0) \cdot W^{\mathbb{Q}^0})_t, \quad 0 \leq t \leq T.$$

Using this, along with (5.9) and the martingale condition (5.8), we compute

$$(5.11) \quad I_t(\mathbb{Q}|\mathbb{Q}^0) = \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{2} \int_t^T \|q_u - q_u^0\|^2 du \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

Now we explicitly consider \mathbb{Q} as a perturbation around \mathbb{Q}^0 . Introduce, for some small parameter ϵ , a parametrised family of measures $\{\mathbb{Q}(\epsilon)\}_{\epsilon \in \mathbb{R}}$, such that

$$(5.12) \quad \mathbb{Q} \equiv \mathbb{Q}(\epsilon), \quad \mathbb{Q}^0 \equiv \mathbb{Q}(0),$$

and also write

$$(5.13) \quad q - q^0 =: -\epsilon\varphi,$$

for some process φ . Then (5.10) becomes

$$(5.14) \quad \sigma\varphi = \mathbf{0}_d.$$

Denote by $\Phi(\mathbf{M}_f)$ the set of such φ which correspond to $\mathbb{Q} \in \mathbf{M}_f$, and also define the process $\Phi := \int_0^\cdot \varphi_s ds$.

The $\mathbb{Q}(\epsilon)$ -dynamics of the state variables S, Y in this notation are then

$$(5.15) \quad dS_t = \text{diag}_d(S_t)\sigma_t dW_t^{\mathbb{Q}(\epsilon)},$$

$$(5.16) \quad dY_t = \text{diag}_{m-d}(Y_t)[(\mu_t^Y - \beta_t q_t^0)dt + \beta_t(dW_t^{\mathbb{Q}(\epsilon)} + \epsilon\varphi_t)].$$

Observe that if we define the state variable $X := (S, Y)^*$, then we have recovered dynamics of the general form (3.2).

The $\mathbb{Q}(\epsilon)$ -dynamics (5.15) of S , along with the constraint (5.14), lead to the following orthogonality result.

Lemma 5.1. *Consider integrands θ^ϵ, φ such that $(\theta^\epsilon \cdot S)$ is a $\mathbb{Q}(\epsilon)$ -martingale and φ satisfies (5.14). Then the stochastic integrals $(\theta^\epsilon \cdot S)$ and $(\varphi \cdot W^{\mathbb{Q}(\epsilon)})$ are orthogonal $\mathbb{Q}(\epsilon)$ -martingales. That is,*

$$\mathbb{E}^{\mathbb{Q}(\epsilon)}[(\theta^\epsilon \cdot S)_T(\varphi \cdot W^{\mathbb{Q}(\epsilon)})_T] = 0.$$

Proof. This is a straightforward computation using (5.15) and (5.14). □

Remark 5.2. Notice that this is precisely the statement of Property 3.1 that was used in the proof of the abstract asymptotic expansion of Theorem 3.2.

Now let F be an \mathcal{F}_T -measurable square-integrable functional of the paths of S, Y , and hence of the Brownian paths, representing the payoff of a European contingent claim. When the dynamics of the state variables are given as in (5.15) and (5.16), we write $F(W^{\mathbb{Q}(\epsilon)} + \epsilon\Phi) \equiv F^\epsilon$. Write the Galtchouk-Kunita-Watanabe decomposition of F^ϵ under $\mathbb{Q}(\epsilon)$ as

$$(5.17) \quad F^\epsilon = \mathbb{E}^{\mathbb{Q}(\epsilon)}[F^\epsilon] + (\theta^\epsilon \cdot S)_T + (\xi^\epsilon \cdot W^{\mathbb{Q}(\epsilon)})_T,$$

for some integrands $\theta^\epsilon, \xi^\epsilon$, such that the stochastic integrals in (5.17) are orthogonal $\mathbb{Q}(\epsilon)$ -martingales, so we have

$$\mathbb{E}^{\mathbb{Q}(\epsilon)}[(\theta^\epsilon \cdot S)_T(\xi^\epsilon \cdot W^{\mathbb{Q}(\epsilon)})_T] = 0.$$

On using (5.11) and (5.13), the indifference price process, as given by Lemma 4.8, then has the stochastic control representation

$$p_t(\alpha) = \sup_{\varphi \in \Phi(\mathbf{M}_f)} \mathbb{E}^{\mathbb{Q}(\epsilon)} \left[F^\epsilon - \frac{\epsilon^2}{2\alpha} \int_t^T \|\varphi_u\|^2 du \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

If we choose

$$(5.18) \quad \epsilon^2 = \alpha,$$

then we get a control problem of the form

$$(5.19) \quad p_t(\alpha) = \sup_{\varphi \in \Phi(\mathbf{M}_f)} \mathbb{E}^{\mathbb{Q}(\epsilon)} \left[F^\epsilon - \frac{1}{2} \int_t^T \|\varphi_u\|^2 du \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

subject to $\mathbb{Q}(\epsilon)$ -dynamics of S, Y given by (5.15), (5.16), and with $\mathbb{Q}(0)$ corresponding to the MEMM \mathbb{Q}^0 . We have now formulated the indifference pricing control problem in the form of the control problem described in Section 3, and applying Theorem 3.2, we have the following result.

Theorem 5.3. *Let the payoff of the claim, F , be a square-integrable functional of the paths of S, Y , having $\mathbb{Q}(\epsilon)$ -dynamics (5.15, 5.16), with $\mathbb{Q}(\epsilon)$ given by (5.12), and with the parameter ϵ given by (5.18). Then for small risk aversion α , the indifference price process of the claim has the asymptotic expansion*

$$(5.20) \quad p_t(\alpha) = \mathbb{E}^{\mathbb{Q}^0}[F|\mathcal{F}_t] + \frac{1}{2}\alpha\mathbb{E}^{\mathbb{Q}^0}\left[\int_t^T \|\xi_u^0\|^2 du \middle| \mathcal{F}_t\right] + O(\alpha^2), \quad 0 \leq t \leq T,$$

where \mathbb{Q}^0 is the minimal entropy martingale measure, and ξ^0 is the process in the Kunita-Watanabe decomposition (5.17) of the claim, under $\mathbb{Q}(0) \equiv \mathbb{Q}^0$.

Proof. This is a direct application of Theorem 3.2 to the stochastic control problem (5.19). □

The underlying message of Theorem 5.3 is that for small risk aversion, the lowest order contribution to the indifference price process is the marginal utility-based price process $\hat{p}_t := \mathbb{E}^{\mathbb{Q}^0}[F|\mathcal{F}_t]$, corresponding to the valuation methodology developed by Davis [6]. The first order correction is a mean-variance correction, since the Kunita-Watanabe decomposition (5.17) for $\epsilon = 0$ is the Föllmer-Schweizer-Sondermann decomposition of the claim under \mathbb{Q}^0 , and the integrand θ^0 in (5.17) is a risk-minimising strategy in the sense of Föllmer and Sondermann [10] under \mathbb{Q}^0 . Similar results have been obtained for bounded claims by Mania and Schweizer [23] and Kallsen and Rheinländer [20]. The contribution here is to show a new methodology for obtaining this result, for a square-integrable claim. The strategy θ^0 is, in general, the zero risk aversion limit of the optimal hedging strategy $\theta(\alpha)$ (see, for example, [23, 20] for a bounded claim), and hence can also be interpreted as the marginal utility-based hedging strategy.

Note that using (5.17) for $\epsilon = 0$, we can write (5.20) as

$$\begin{aligned} p_t(\alpha) &= \mathbb{E}^{\mathbb{Q}^0}[F|\mathcal{F}_t] \\ &+ \frac{1}{2}\alpha \left(\text{var}^{\mathbb{Q}^0}[F|\mathcal{F}_t] - \mathbb{E}^{\mathbb{Q}^0}\left[\int_t^T \|\theta_u^0\|^2 d[S]_u \middle| \mathcal{F}_t\right] \right) + O(\alpha^2), \end{aligned}$$

for $t \in [0, T]$, which highlights the mean-variance structure of the asymptotic representation.

6. APPLICATIONS

Here we show some examples where Theorem 5.3 would apply.

Example 6.1 (Multi-dimensional basis risk model). This is the model of Section 5, with d traded stocks S and $(m - d)$ non-traded assets Y , and with the volatility process σ in (5.1) given by

$$\sigma_t = \begin{pmatrix} \sigma_t^S & \mathbf{0}_{d \times (m-d)} \end{pmatrix}, \quad 0 \leq t \leq T,$$

where $\mathbf{0}_{d \times (m-d)}$ denotes the zero $d \times (m - d)$ matrix. Write the m -dimensional Brownian motion W as $W = (W^S, W^{S,\perp})^*$, where W^S denotes the first d components of W . Then the d traded stocks are driven by d Brownian motions, and the non-traded assets are

imperfectly correlated with S . The claim payoff F is typically dependent on the evolution of Y only, though our results are valid without this restriction.

In this case, the process λ in (5.2) and the integrand q in (5.3) are given by

$$\lambda_t = \begin{pmatrix} \lambda_t^S \\ \mathbf{0}_{m-d} \end{pmatrix}, \quad q_t = \begin{pmatrix} \lambda_t^S \\ \gamma_t \end{pmatrix}, \quad 0 \leq t \leq T,$$

where λ^S is the stock's d -dimensional market price of risk process, given by $\lambda^S := \sigma^{S*}(\sigma^S \sigma^{S*})^{-1} \mu^S$, and γ is an $(m-d)$ -dimensional adapted process. Each choice of γ leads to a different ELMM \mathbb{Q} , with $\gamma = \mathbf{0}_{m-d}$ corresponding to the minimal martingale measure \mathbb{Q}_M . Then the density process of any ELMM $\mathbb{Q} \in \mathbf{M}_f$ is given by

$$Z_t^{\mathbb{Q}} = \mathcal{E}(-\lambda^S \cdot W^S - \gamma \cdot W^{S,\perp})_t, \quad 0 \leq t \leq T.$$

A special feature of these models arises when the process λ^S is either deterministic or does not depend on the non-traded asset prices Y . In this case it is not hard to see that the MEMM $\mathbb{Q}^0 = \mathbb{Q}_M$. This is because the relative entropy process between $\mathbb{Q} \in \mathbf{M}_f$ and \mathbb{P} is given by

$$(6.1) \quad I_t(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{2} \int_t^T (\|\lambda_u^S\|^2 + \|\gamma_u\|^2) du \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

The problem of finding the minimal entropy martingale measure is then to minimise this functional subject to \mathbb{Q} -dynamics of S, Y given by (5.6, 5.7). The process γ plays the role of a control. In the current notation, the \mathbb{Q} -dynamics of Y are given as

$$dY_t = \text{diag}_{m-d}(Y_t) \left[\left(\mu_t^Y - \beta_t \begin{pmatrix} \lambda_t^S \\ \gamma_t \end{pmatrix} \right) dt + \beta_t dW_t^{\mathbb{Q}} \right].$$

From this it is clear that if λ^S does not depend on Y , then it is unaffected by the control, and then the relative entropy process in (6.1) is minimised by choosing $\gamma = \mathbf{0}_{m-d}$, so $\mathbb{Q}_E \equiv \mathbb{Q}^0 = \mathbb{Q}_M$. In this case, the Kunita-Watanabe decomposition of the claim under \mathbb{Q}^M will be of the form

$$(6.2) \quad F = \mathbb{E}^{\mathbb{Q}^M}[F] + (\theta^M \cdot S)_T + (\xi^M \cdot W^{\mathbb{Q}^M})_T,$$

for integrands θ^M, ξ^M such that the \mathbb{Q}_M -martingales $(\theta^M \cdot S)$ and $(\xi^M \cdot W^{\mathbb{Q}^M})$ are orthogonal, and $W^{\mathbb{Q}^M}$ is a \mathbb{Q}_M -Brownian motion. An example where this pertains is given in Monoyios [28], in a two-dimensional model of basis risk with partial information. The indifference price expansion at time zero is then given as

$$(6.3) \quad p_0(\alpha) = \mathbb{E}^{\mathbb{Q}^M}[F] + \frac{1}{2} \alpha \left(\text{var}^{\mathbb{Q}^M}[F] - \mathbb{E}^{\mathbb{Q}^M} \left[\int_0^T \|\theta_t^M\|^2 d[S]_t \right] \right) + O(\alpha^2).$$

When the model is Markovian, the integrand θ^M can sometimes be expressed in terms of the partial derivatives with respect to S and Y of the marginal price process $\hat{p}(t, S_t, Y_t) = \mathbb{E}^{\mathbb{Q}^M}[F|S_t, Y_t]$.

Example 6.2 (Lognormal basis risk model). This is the classical example first considered by Davis [7], and is a special case of Example 6.1, with constant parameters, so once again we will have $\mathbb{Q}^0 = \mathbb{Q}^M$. Here we show how the asymptotic expansion for the indifference price simplifies in this case, and so we extend results of Davis and others [7, 15, 24] to general (and so possibly path-dependent) payoffs dependent on the non-traded asset price.

Set $d = 1$, $m = 2$ in Example 6.1, and set $\mu^S, \mu^Y, \sigma^S, \beta$ to be constant, with

$$\beta = \sigma^Y \begin{pmatrix} \rho & \sqrt{1-\rho^2} \end{pmatrix}, \quad \rho \in (-1, 1),$$

for constants σ^Y, ρ . Then the stock and non-traded asset are imperfectly correlated with cross-variation process given by

$$[S, Y]_t = \rho \sigma^S \sigma^Y \int_0^t S_u Y_u du, \quad 0 \leq t \leq T.$$

The \mathbb{P} -dynamics of the assets are

$$dS_t = \sigma^S S_t (\lambda^S dt + dW_t^S), \quad dY_t = Y_t (\mu^Y dt + \sigma^Y dW_t^Y), \quad \lambda^S := \mu^S / \sigma^S,$$

where $W^Y = \rho W^S + \sqrt{1 - \rho^2} W^{S, \perp}$. It is easy to see that in this case $\mathbb{Q}^0 = \mathbb{Q}_M$, under which the dynamics of the asset prices are

$$(6.4) \quad dS_t = \sigma^S S_t dW_t^{S, \mathbb{Q}_M}, \quad dY_t = Y_t [(\mu^Y - \rho \lambda^S \sigma^Y) dt + \sigma^Y dW_t^{Y, \mathbb{Q}_M}],$$

for \mathbb{Q}_M -Brownian motions $W^{S, \mathbb{Q}_M}, W^{Y, \mathbb{Q}_M}$ with instantaneous correlation ρ , so that

$$(6.5) \quad W^{Y, \mathbb{Q}_M} = \rho W^{S, \mathbb{Q}_M} + \sqrt{1 - \rho^2} W^{S, \perp, \mathbb{Q}_M},$$

with $W^{S, \mathbb{Q}_M}, W^{S, \perp, \mathbb{Q}_M}$ independent \mathbb{Q}_M -Brownian motions.

Suppose that the claim payoff F depends only on the evolution of Y (but can indeed be path-dependent). Then the Kunita-Watanabe decomposition of F under \mathbb{Q}^M will be of the special form

$$(6.6) \quad F = \mathbb{E}^{\mathbb{Q}_M}[F] + (\psi \cdot W^{Y, \mathbb{Q}_M})_T,$$

for some process ψ such that $(\psi \cdot W^{Y, \mathbb{Q}_M})$ is a \mathbb{Q}^M -martingale. But we also have the general form of this decomposition, given by the analogue of (6.2), which in this case reads as

$$(6.7) \quad F = \mathbb{E}^{\mathbb{Q}_M}[F] + (\theta^M \cdot S)_T + (\xi^M \cdot W^{S, \perp, \mathbb{Q}_M})_T,$$

for integrands θ^M, ξ^M (here, θ^M would be the marginal utility-based hedging strategy, or equivalently, the risk-minimising strategy for the claim).

Equating the representations in (6.6) and (6.7) and in view of (6.4) and (6.5), it is easy to see that θ^M, ξ^M are both linearly related to the process ψ , through

$$\theta^M \sigma^S S = \rho \psi, \quad \xi^M = \sqrt{1 - \rho^2} \psi.$$

It is then straightforward to compute that

$$\text{var}^{\mathbb{Q}_M}[F] = \frac{1}{\rho^2} \mathbb{E}^{\mathbb{Q}_M} \left[\int_0^T (\theta_t^M)^2 d[S]_t \right].$$

The time-zero indifference price expansion (6.3) in this case then simplifies to

$$p_0(\alpha) = \mathbb{E}^{\mathbb{Q}_M}[F] + \frac{1}{2} \alpha (1 - \rho^2) \text{var}^{\mathbb{Q}_M}[F] + O(\alpha^2),$$

which is an extension of the form found in [7, 15, 24] to general square-integrable European payoffs.

Example 6.3 (Basis risk with stochastic correlation). This model has been considered by Ankirchner and Heyne [1], who examined local risk minimisation methods for hedging basis risk. A traded asset S and non-traded asset Y follow correlated geometric Brownian motions, as in Example 6.2, but the correlation $\rho = (\rho_t)_{0 \leq t \leq T}$ is now stochastic. In this case, we have $m = 3$, $d = 1$, and with W a three-dimensional Brownian motion, let

$W^S = W^1$, $W^Y = \rho W^1 + \sqrt{1 - \rho^2} W^2$, $W^\rho = \delta W^1 + \eta W^2 + \sqrt{1 - \delta^2 - \eta^2} W^3$, for constants δ, η such that $\delta^2 + \eta^2 \leq 1$. The state variable dynamics are then

$$\begin{aligned} dS_t &= \sigma^S S_t (\lambda^S dt + dW_t^1), \\ dY_t &= Y_t [\mu^Y dt + \sigma^Y (\rho_t dW_t^1 + \sqrt{1 - \rho^2} dW_t^2)], \\ d\rho_t &= g_t dt + h_t (\delta dW_t^1 + \eta dW_t^2 + \sqrt{1 - \delta^2 - \eta^2} dW_t^3). \end{aligned}$$

Here, g, h are adapted processes such that $\rho_t \in [-1, 1]$ almost surely. Ankirchner and Heyne [1] give some specific examples of such models.

In this example we also have $\mathbb{Q}^0 = \mathbb{Q}_M$, with $Z^{\mathbb{Q}_M} = \mathcal{E}(-\lambda^S W^1)$, and the Föllmer-Schweizer-Sondermann decomposition of the claim is of the form

$$(6.8) \quad F = \mathbb{E}^{\mathbb{Q}_M}[F] + (\theta^M \cdot S)_T + (\xi^M \cdot W^{2, \mathbb{Q}_M})_T + (\phi^M \cdot W^{3, \mathbb{Q}_M})_T,$$

for some integrands θ^M, ξ^M, ϕ^M , and where $W^{\mathbb{Q}_M} = (W^{1, \mathbb{Q}_M}, W^{2, \mathbb{Q}_M}, W^{3, \mathbb{Q}_M})^*$ is a three-dimensional \mathbb{Q}_M -Brownian motion, the first of which drives the stock, so that the stochastic integrals in (6.8) are orthogonal \mathbb{Q}_M -martingales. The time-zero indifference price expansion is again of the form (6.3).

Many other examples are covered by the framework of Theorem 5.3, including classical stochastic volatility models, which are similar in some ways to the two-dimensional basis risk models, and basis risk models with stochastic volatility (so $m = 3$, $d = 1$) with a traded and non-traded asset both driven by a common stochastic volatility process (and stochastic correlation can be added to this framework). An interesting case would be an extension of the model considered in [28], a basis risk model with unknown asset drifts. In [28] the unknown drifts are modelled as unknown constants, and hence as random variables with a prior distribution, which is updated via a Kalman filter. It would be interesting to model the drifts as linear diffusions, which would result in a random parameter basis risk model.

6.1. Entropy minimisation in stochastic volatility models. We end with another application of the asymptotic methods developed in the paper. This time, we are interested in finding the minimal entropy martingale measure $\mathbb{Q}^0 \equiv \mathbb{Q}_E$ in a stochastic volatility model. A traded asset S and a non-traded stochastic factor Y follow, under the physical measure \mathbb{P} ,

$$(6.9) \quad dS_t = \sigma(Y_t) S_t (\lambda(Y_t) dt + dW_t^S),$$

$$(6.10) \quad dY_t = a(Y_t) dt + b(Y_t) dW_t^Y,$$

for suitable functions σ, λ, a, b such that there are unique strong solutions to (6.9, 6.10). The Brownian motions W^S, W^Y have constant correlation $\rho \in [-1, 1]$. We write $W_t^Y = \rho W_t^S + \sqrt{1 - \rho^2} W_t^{S, \perp}$. The density process of any ELMM \mathbb{Q} is

$$Z_t^{\mathbb{Q}} = \mathcal{E}(-\lambda \cdot W^S - \gamma \cdot W^{S, \perp})_t, \quad 0 \leq t \leq T,$$

for some square-integrable process γ such that $Z^{\mathbb{Q}}$ is a \mathbb{P} -martingale.

The entropy minimisation problem is the stochastic control problem to minimise

$$I_0(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{2} \int_0^T (\lambda^2(Y_t) + \gamma_t^2) dt \right],$$

over control processes γ , where we assume that $I_0(\mathbb{Q}|\mathbb{P}) < \infty$, and where, under \mathbb{Q} , S, Y follow

$$\begin{aligned} dS_t &= \sigma(Y_t)S_t dW_t^{S,\mathbb{Q}}, \\ (6.11) \quad dY_t &= (a(Y_t) - b(Y_t)\rho\lambda(Y_t))dt + b(Y_t)(dW_t^{Y,\mathbb{Q}} - \sqrt{1-\rho^2}\gamma_t dt), \end{aligned}$$

for \mathbb{Q} -Brownian motions $W^{S,\mathbb{Q}}, W^{Y,\mathbb{Q}}$ with correlation ρ , such that setting $\gamma = 0$ yields the minimal martingale measure \mathbb{Q}_M .

The idea here is to consider the drift adjustment $\sqrt{1-\rho^2}\gamma_t$ in (6.11) as a perturbation to the Brownian paths, and hence to convert the entropy minimisation problem to the type of control problem we have considered in Section 3, in the limit that the absolute value of the correlation is close to 1, so $1-\rho^2$ is small. To this end, we define a parameter ϵ and a control process φ such that

$$\epsilon^2 = 1 - \rho^2, \quad \epsilon\varphi = -\sqrt{1-\rho^2}\gamma,$$

and we define a parametrised family of measures $\{\mathbb{Q}(\epsilon)\}_{\epsilon \in \mathbb{R}}$, such that

$$\mathbb{Q} = \mathbb{Q}(\epsilon), \quad \mathbb{Q}(0) = \mathbb{Q}_M.$$

The state variable dynamics for Y are then given by

$$(6.12) \quad dY_t = (a(Y_t) - b(Y_t)\rho\lambda(Y_t))dt + b(Y_t)(dW_t^{Y,\mathbb{Q}(\epsilon)} + \epsilon\varphi_t dt).$$

With $\Phi := \int_0^\cdot \varphi_s ds$, we define a square-integrable functional $F^\epsilon \equiv F(W^{\mathbb{Q}(\epsilon)} + \epsilon\Phi)$ of the Brownian paths by

$$F^\epsilon := \frac{1}{2} \int_0^T \lambda^2(Y_t) dt =: \frac{1}{2} K_T,$$

where, for brevity of notation, we have defined the so-called mean-variance trade-off process K by

$$(6.13) \quad K_t := \int_0^t \lambda^2(Y_u) du, \quad 0 \leq t \leq T.$$

In this notation, the relative entropy between the minimal martingale measure and \mathbb{P} is

$$(6.14) \quad I_0(\mathbb{Q}_M|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}_M} \left[\frac{1}{2} K_T \right] = \mathbb{E}^{\mathbb{Q}(0)}[F^0].$$

The control problem to minimise $I_0(\mathbb{Q}|\mathbb{P})$ over ELMs $\mathbb{Q} \in \mathbf{M}_f$ has value function

$$(6.15) \quad I_0(\mathbb{Q}_E|\mathbb{P}) := \inf_{\varphi \in \Phi(\mathbf{M}_f)} \mathbb{E}^{\mathbb{Q}(\epsilon)} \left[F^\epsilon + \frac{1}{2} \int_0^T \varphi_t^2 dt \right],$$

where $\Phi(\mathbf{M}_f)$ denotes the set of controls φ such that $I_0(\mathbb{Q}|\mathbb{P})$ is finite.

We have now formulated the entropy minimisation problem in the form we need to be able to apply the Malliavin asymptotic method, and this gives the theorem below.

Theorem 6.4. *In the stochastic volatility model defined by (6.9,6.10), the relative entropy between the minimal entropy martingale measure \mathbb{Q}_E and \mathbb{P} , in the limit that $1-\rho^2$ is close to 1, is given as*

$$I_0(\mathbb{Q}_E|\mathbb{P}) = I_0(\mathbb{Q}_M|\mathbb{P}) - \frac{1}{8}(1-\rho^2)\text{var}^{\mathbb{Q}_M}[K_T] + O((1-\rho^2)^2),$$

where \mathbb{Q}_M is the minimal martingale measure and K is the mean-variance trade-off process defined in (6.13).

Proof. This is along the same lines as previous results, so we only sketch the details. One appeals to the decomposition of F^ϵ under $\mathbb{Q}(\epsilon)$, which is of the form

$$(6.16) \quad F^\epsilon = \mathbb{E}^{\mathbb{Q}(\epsilon)}[F^\epsilon] + (\xi^\epsilon \cdot W^{\mathbb{Q}(\epsilon)})_T,$$

for some integrand ξ^ϵ . Such a decomposition exists uniquely, given that F^ϵ depends only on Y , and the dynamics in (6.12). In particular, this decomposition also holds for $\epsilon = 0$. We expand the objective function (6.15) about $\epsilon = 0$ and use the representation (6.16) for $\epsilon = 0$. This gives

$$\mathbb{E}^{\mathbb{Q}(\epsilon)} \left[F^\epsilon + \frac{1}{2} \int_0^T \varphi_t^2 dt \right] = \mathbb{E}^{\mathbb{Q}(0)} \left[F^0 + \int_0^T \left(\epsilon \xi_t^0 \varphi_t + \frac{1}{2} \varphi_t^2 \right) dt \right] + o(\epsilon)$$

We minimise the right-hand-side over φ by choosing $\varphi = -\epsilon \xi^0$. Using (6.16) at $\epsilon = 0$ again, and recalling (6.14), the result follows. \square

Remark 6.5. In [26, 25], Esscher transform relations between \mathbb{Q}_E and \mathbb{Q}_M are derived, and it is an exercise in asymptotic analysis to see that those results are consistent with Theorem 6.4.

7. CONCLUSIONS

It is quite natural to apply Malliavin calculus ideas in stochastic control problems where the control turns out to be a drift which is considered as a perturbation to a Brownian motion, and this is the path taken in this paper. We have shown how the method can yield small risk aversion asymptotic expansions for exponential indifference prices in Itô process models, and how one can identify the minimal entropy measure as a perturbation to the minimal martingale measure in stochastic volatility models. It would be interesting to extend the method to models with jumps in the underlying state process.

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MICHAEL MONOYIOS, MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, 24–29 ST GILES', OXFORD OX1 3LB, UK

E-mail address: monoyios@maths.ox.ac.uk